Nassau County Interscholastic Math League

Replacement Problem

The hypotenuse of a right triangle is one less than the sum of its legs. A second right triangle has legs each one less than the corresponding leg of the first right triangle. Find the numerical value of the area of this second right triangle.

Answer: $\frac{1}{4}$

Solution: Let the legs of the first be a and b, hypotenuse a + b – 1.
Note that the legs of the second right triangle will be a – 1 and b – 1.

From the Pythagorean Theorem,

$$a^2 + b^2 = (a + b - 1)^2$$

Now, let's do some algebra

$$a^2 + b^2 = a^2 + b^2 + 1 + 2ab - 2a - 2b$$
$$0 = 1 - 2a - 2b + 2ab$$
$$1 = 2 - 2a - 2b + 2ab$$
$$\frac{1}{2} = 1 - a - b + ab$$
$$\frac{1}{2} = (1-a)(1-b) = (a-1)(b-1)$$

The area of the second triangle is $\frac{1}{2}(a-1)(b-1) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$