Nassau County Interscholastic Mathematics League

Solutions **2002-2003**

Contest # 4

T1. Apply the angle bisector proportion to ΔBAD and ΔCAE. So we get that AB: AD = BC:CD = 1:4 and AC:AE = CD:DE = 3:5. Rewrite so BC:CD = 3:12 and CD:DE = 12:20, so BC:CD:DE = 3:12:20

T2. Since the three numbers all leave the same remainder on division by n, the difference of any pair of the numbers must be divisible by n. The differences are 96, 240, and 336. The greatest common divisor is 48, so that is the largest possible value for n.

T3. A googol = 10^{100} so its square = 10^{200}, which is a 1 followed by 200 zeroes. So 3 down is 201. Now the only four digit perfect cubes with third digit 2 or 1 are 1728 and 4913, so 1 Across is 1728 and 6 Across is 4913. The only Fibonacci number in the 100's is 144, so that's 1 Down. Of the squares in the 4000's only 4900 has a 0 in the third digit, so 2 Across is 4900. Finally, check that all digits of 2 Down are odd and 803 = (11)(73).

T4. P(matching pair) = $\frac{{}_3C_2 + {}_5C_2 + {}_4C_2}{{}_{12}C_2} = \frac{19}{66}$ and the probability of not matching is thus $1 - \frac{19}{66} = \frac{47}{66}$. Now the probability that at least one day they match is one minus the probability that they don't match all five days. So the result is $1 - \left(\frac{47}{66}\right)^5 \approx 0.817$ [aren't you glad to have a calculator?]

T5. Look at the sequence of functions $f_1(x) = \frac{x-3}{x+1}, f_2(x) = f \circ f_1(x) = \frac{x+3}{1-x}, \quad f_3(x) = f \circ f_2(x) = x$. This means that every third function is the identity function, i.e. $f_3(x) = f_6(x) = \cdots = f_{2001}(x) = x$. So $f_{2002}(x) = f_1(x) = \frac{x-3}{x+1}$ and $f_{2002}(4) = \frac{1}{5}$

T6. Since the number is divisible by 99, it will be divisible by both 9 and 11. So $11 + a + b$ is divisible by 9, and $11 + a + b = 18$ or 27, so $a + b = 7$ or 16. Also $(3+2+4)-(2+a+b)$ is divisible by 11, so $7 - (a+b) = 0$ or -11 and a+b = 7 or 18. Since it must work for both, a+b = 7.